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Construction of diffusion processes penetrating fractals -An application of the theory of Besov spaces-

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1 Introduction

Assume that there are countable number of disordered media $\{K_i\}_{i=1}^{M}$ $(1 \leq M \leq \infty)$ on $\mathbf{R}^N$. Can we construct a diffusion process which moves the whole space, whose behaviour is like Brownian motion on K_i for each media and like Brownian motion on $\mathbf{R}^N$ outside? If we can, how does the diffusion behave asymptotically? In this paper, we will treat this problem when K_i's are fractals.

Since late 80's, there have been many works for diffusion processes and Laplace operators on fractals (see [1], [9], [11] e.t.c.). Recent works ([6], [7], [10]) reveal that domains of the corresponding quadratic forms (Dirichlet forms) are Besov spaces and that theories of Besov spaces could be applied to this field. Our work shows that trace theory of Besov spaces is applicable to the question posed.

The initial work on diffusion processes penetrating fractals was by Lindstrøm [14] and has been followed up by more general constructions in [7], [10]. These

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papers have been primarily interested in demonstrating the existence of a process which behaves like a diffusion on a fractal within a subset of Euclidean space, yet standard Brownian motion outside. Our work will extend this construction to incorporate many different fractals which may be embedded in some Euclidean space (Figure 2), but also may tile the space (Figure 1). We will call spaces of either type *fractal fields*.

A key example that we would like the reader to bear in mind throughout the paper is the gasket tiling in $\mathbf{R}^2$. Consider a triangular lattice on $\mathbf{R}^2$ where each edge is of length 1. We will fill each triangle with a version of the Sierpinski gasket in a periodic way. More precisely, let $SG(l)$ be the family of 2-dimensional Sierpinski gaskets from [3] with sidelength 1 constructed by contraction maps with contraction factor $1/l$. Now, take a set of triangles (we let L be the number of triangles in the set) from the triangular lattice so that the union of them is a connected closed set. In each triangle we place $\{SG(l_k)\}_{k=1}^{L}$ and denote the union of these fractals by K_0. Without loss of generality, we can assume that one of the vertices of K_0 is $(0,0)$. We take $i_x \in \mathbf{N}$ so that $K_0 \cap (K_0 + (i_x, 0)) \neq \emptyset$ and $\text{Int } K_0 \cap \text{Int } (K_0 + (i_x, 0)) = \emptyset$. We also take $i_y \in \mathbf{N}$ in the same way by taking $(0, i_y)$ instead of $(i_x, 0)$. Then, $G \equiv \cup_{l,m \in \mathbf{Z}} (K_0 + (l i_x, m i_y))$ is the space we will consider. Figure 1 indicates the case when K_0 is a parallelogram filled with $SG(2)$ and $SG(4)$.

This paper will treat the general construction problem. We incorporate the trace theory of Besov spaces, for the embedding into a Euclidean space, with an idea originally due to Kusuoka, [12] which shows how to extend a Lipschitz function from the boundary of a fractal to the interior while controlling its energy. This will allow us to build up a Dirichlet form and establish some properties, such as a Nash inequality. In the forthcoming paper [5], we will further discuss on heat kernel estimates and the large deviations of our diffusion process. In the paper, we will demonstrate the shape of the shortest paths through our fractal fields and observe that it is fractals with small d_w which take the longest to cross (in the short time limit) and this allows us to determine the shortest paths in a recursive manner, first fixing them through the slow parts and filling in the details for the faster paths.

2 Fractal fields and their Dirichlet forms

In this section we will introduce fractal fields, the framework within which we will work. Our aim is to construct local regular Dirichlet forms on these spaces. Let $\{K_i\}_{i=1}^M \subset \mathbf{R}^2$ $(1 \le M \le \infty)$ be a family of (bounded or unbounded) nested fractals whose definition will be given in Appendix. When K_i is unbounded, we denote by $\hat{K}_i$ the corresponding bounded nested fractal (when K_i is bounded, $\hat{K}_i = K_i$) and denote by $\{\Psi_j^{(i)}\}_{j\in S_i}$ the family of contractions which determine $\hat{K}_i$ $(S_i = \{1, 2, \cdots, N_i\})$. Let $V_0^{(i)}$ be the set of essential fixed points for $\hat{K}_i$.

For each closed set A, let Cov (A) be the set of points covered by A, i.e., decomposing $\mathbf{R}^2 \setminus A$ into connected components $\{D_j\}_{j=1}^\infty$ and denoting by $\{D_j\}_{j\in U(A)}$ the unbounded components, Cov $(A) = \mathbf{R}^2 \setminus \cup_{j\in U(A)} D_j$. We note that if the set A has holes, these may be contained in Cov(A). We assume the following for the location of $\{K_i\}_i$.

Assumption 2.1 *1) For each* $1 \le i \ne j \le M$,

$$Int\ (Cov\ (K_i)) \cap Int\ (Cov\ (K_j)) = \emptyset,$$

where $Int\ (K)$ *is the interior of* K.

2) For each compact set $C \subset \mathbf{R}^2$,

$$\sharp\{i : C \cap K_i \neq \emptyset\} < \infty.$$

Define $G = \cup_{i=1}^M K_i$ and $D = \mathbf{R}^2 \setminus \mathrm{Cov}(G)$, then G is a closed set by 2) of Assumption 2.1. Clearly, $D = \cup_{j \in U(G)} D_j$. We define $\tilde{G} = G \cup D$ and call it a *fractal field* generated by $\{K_i\}_{i=1}^M$. See Figure 1 and Figure 2 for examples of fractal fields. Note that we can define fractal fields on $\mathbf{R}^N$ in the same way using nested fractals on $\mathbf{R}^N$, but as our Assumption 2.2, which will be introduced later, seldom holds for nested fractals on $N \geq 3$, we will restrict to $N = 2$.

Let $\partial_e G$ be the topological boundary of G as a subset of $\mathbf{R}^2$. For $1 \leq i \neq j \leq M$, let

$$\Gamma_{ij} = \mathrm{Cov}\ (K_i) \cap \mathrm{Cov}\ (K_j), \qquad \partial_i G = \cup_{1 \leq i \neq j \leq M} \Gamma_{ij}. \tag{2.1}$$

Set $\partial G = \partial_e G \cup \partial_i G$. Let μ_i be normalized Hausdorff measure on K_i, i.e. $\mu_i(\hat{K}_i) = 1$, and set $\mu = \sum_{i=1}^M \mu_i$, $\tilde{\mu} = m|_D + \mu$ where m is the Lebesgue measure on $\mathbf{R}^2$.

We next define a form on $\tilde{G}$. First, for each i, the local regular Dirichlet form $(\mathcal{E}_{K_i}, \mathcal{F}_{K_i})$ on $\mathbf{L}^2(K_i, \mu_i)$ is given as in Theorem A.2 and Theorem A.5. We denote $d_f(K_i), d_s(K_i), d_w(K_i)$ the Hausdorff, spectral and walk dimension respectively w.r.t. Euclidean metric. Let $K \subset K_i$ be a compact nested fractal which is congruent to $\hat{K}_i$ (thus, when K_i is bounded, $K = K_i$). For each Γ_{ij} in (2.1) where $1 \leq i \neq j \leq M$ and for $\omega \in \Sigma_i \equiv (S_i)^{\mathbf{N}}$, let $d_{\Gamma_{ij},K}(\omega) = \min\{n \geq 1 : \Gamma_{ij} \cap \Psi^{(K)}_{\omega_1 \cdots \omega_n}(K) = \emptyset\}$ where $\{\Psi_j^{(K)}\}_{j \in S_i}$ is a family of α_i-contractions which determine K, and define

$$\kappa(\Gamma_{ij}, K) = -\frac{1}{\log N_i} \limsup_{n \to \infty} \frac{1}{n} \log \nu_i(d_{\Gamma_{ij},K}(\omega) > n),$$

where ν_i is a Bernoulli measure on Σ_i so that $\nu_i(\{\omega \in \Sigma_i : \omega_1 = l\}) = 1/N_i$ for each $l \in S_i$. We adopt the convention that $-\log 0 = \infty$.

Assumption 2.2 *For each $1 \leq i \neq j \leq M$, the following holds where K and Γ_{ij} are as above,*

$$\frac{2}{d_s(K)} - \frac{2}{d_f(K)} < \kappa(\Gamma_{ij}, K). \tag{2.2}$$

Remark 2.3 *For the gasket tiling introduced in the Introduction (also indicated in Figure 1), (2.2) always holds. Indeed, let $K = SG(l)$ $l \geq 2$ and $\Gamma = \Gamma_{ij}$ be the bottom line of K. As there are l^n n-cells which intersect with Γ, we see that $\nu(d_{\Gamma,K}(\omega) > n) = l^n/L^n$ where $L = l(l+1)/2$. Thus, $\kappa(\Gamma, K) = 1 - \log l/\log L$ and (2.2) is equivalent to*

$$\frac{\log(\rho L) - 2\log l}{\log L} < 1 - \frac{\log l}{\log L},$$

which is equivalent to $\rho < l$. Note that $\rho = P^{x_0}(\tau_{V_0\setminus\{x_0\}}(X) < \tau_{x_0}(X))^{-1}$ where $x_0 \in V_0$, X is a Markov chain corresponding to $(\mathcal{E}_{SG(l)})_1$, and $\tau_A(X)$ is the first hitting time of X to A. Note also that if we define $\bar{X}$ be a simple random walk on $\mathbf{Z}$, then $l = P^0(\tau_{\{-l,l\}}(\bar{X}) < \inf\{n \geq 1 : \bar{X}(n) = 0\})^{-1}$. Then, by the comparison of escape probabilities using the electrical network method (we use so called cutting law), we can easily obtain $\rho < l$.

Assumption 2.1 and Assumption 2.2 will hold throughout the paper. We define a bilinear form $(\tilde{\mathcal{E}}, \mathcal{D}(\tilde{\mathcal{E}}))$ on $\mathbf{L}^2(\tilde{G}, \tilde{\mu})$ as follows,

$$\begin{aligned}
\tilde{\mathcal{E}}(u,v) &= \sum_{i=1}^{M} \mathcal{E}_{K_i}(u|_{K_i}, v|_{K_i}) + \frac{1}{2}\sum_{j\in U(G)} \int_{D_j} \nabla u(x)\nabla v(x)dx \quad \text{for all } u,v \in \mathcal{D}(\tilde{\mathcal{E}}),\\
\mathcal{D}(\tilde{\mathcal{E}}) &= \{u \in C_0(\tilde{G}) : u|_{K_i} \in \mathcal{F}_{K_i}\ \forall\, i,\ \ u|_{D_j} \in W^{1,2}(D_j)\ \forall\, j,\ \ \tilde{\mathcal{E}}(u,u) < \infty\},
\end{aligned}$$

where $D = \cup_{j\in U(G)} D_j$ is a decomposition of D into open connected components and $C_0(\tilde{G})$ is a space of continuous functions on $\tilde{G}$ with compact support. Then, it is easy to check the following.

Lemma 2.4 *1) $(\tilde{\mathcal{E}}, \mathcal{D}(\tilde{\mathcal{E}}))$ is closable in $\mathbf{L}^2(\tilde{G}, \tilde{\mu})$.*

2) $\mathcal{D}(\tilde{\mathcal{E}})$ is an algebra.

3) For each j, $x \in K_j$ and each $U(x)$ which is a neighborhood of x, there exists $f \in \mathcal{F}_{K_j} \cap C_0(K_j)$ such that $f(x) > 0$ and Supp $f \subset U(x) \cap K_j$ where Supp f denotes the support of f.

4) $C_0^\infty(D) \subset \mathcal{D}(\tilde{\mathcal{E}})$.

Now, denote $\tilde{\mathcal{F}} = \overline{\mathcal{D}(\tilde{\mathcal{E}})}^{\tilde{\mathcal{E}}_{(1)}}$ so that $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ is the smallest extension of $(\tilde{\mathcal{E}}, \mathcal{D}(\tilde{\mathcal{E}}))$, where $\tilde{\mathcal{E}}_{(1)}(f,f) = \tilde{\mathcal{E}}(f,f) + \|f\|^2_{\mathbf{L}^2(\tilde{G},\tilde{\mu})}$. We then have the following.

Theorem 2.5 *$(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ is a local regular Dirichlet form on $\mathbf{L}^2(\tilde{G}, \tilde{\mu})$.*

By the general theory ([2]), there is a one to one correspondence between a local regular Dirichlet form on $\mathbf{L}^2(\tilde{G}, \tilde{\mu})$ and a $\tilde{\mu}$-symmetric diffusion process on $\tilde{G}$ except for some exceptional set of starting points. We will denote by $\{\tilde{X}_t\}_{t\geq 0}$ the diffusion process corresponding to $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$. Note that, as the original forms on $\{K_i\}_i$ and $\{D_j\}_j$ are strong local, $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ is also strong local.

For the proof of Theorem 2.5, the key part is to prove the following.

Proposition 2.6 *1) For each $x \neq y \in \tilde{G}$, there exists $g \in \mathcal{D}(\tilde{\mathcal{E}})$ such that $g(x) \neq g(y)$.*

2) For any compact set L in $\tilde{G}$, there exists $f \in \mathcal{D}(\tilde{\mathcal{E}})$ such that $f|_L = 1$.

Indeed, using this proposition, we can prove Thorem 2.5 as follows. It is easy to see that $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ is a local Dirichlet form. Also, as $\tilde{\mathcal{F}} = \overline{\mathcal{D}(\tilde{\mathcal{E}})}^{\tilde{\mathcal{E}}_{(1)}}$, it is clear that $\mathcal{D}(\tilde{\mathcal{E}})$ is dense in $\tilde{\mathcal{F}}$ w.r.t. $\tilde{\mathcal{E}}_{(1)}$-norm. Thus, all we need for the regularity of the form is to show that $\mathcal{D}(\tilde{\mathcal{E}})$ is dense in $C_0(K)$ w.r.t. $||\cdot||_\infty$-norm. Now, as $\mathcal{D}(\tilde{\mathcal{E}})$ is an algebra (Lemma 2.4 2)), we see that for each compact set L in $\tilde{G}$, $\mathcal{D}(\tilde{\mathcal{E}})|_L$ is dense in $C(L)$ by using Proposition 2.6 and applying the Stone-Weierstrass theorem. This establishes regularity and we have completed the proof.

For each $B \subset \mathbf{R}^2$, define $\tau_B = \inf\{t \geq 0 : \tilde{X}_t \in B\}$. We can then prove that $\tilde{X}_t$ penetrates into each K_i. To say more exactly, we have the following.

Proposition 2.7 *Assume that $m(G) = 0$ where m is the Lebesgue measure on $\mathbf{R}^2$. Then, for any nearly Borel set B with positive 1-capacity (w.r.t. $\tilde{\mathcal{E}}$),*

$$\tilde{P}^x(\tau_B < \infty) > 0 \qquad \text{for quasi-every } x \in \mathbf{R}^2. \tag{2.3}$$

Especially, when B is either a subset of K_i whose 1-capacity w.r.t. $\mathcal{E}_{K_i}$ is positive or a subset of $\mathbf{R}^2$ whose 1-capacity w.r.t. the Dirichlet integral is positive, then (2.3) holds.

The proof is the same as Proposition 2.9 in [10].

In the same way as Theorem 2.11 in [10], we can prove a Nash type estimate for the heat semigroup. Let $P_t^{\tilde{\mathcal{E}}}$ $(t > 0)$ be the semigroup corresponding to $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$. Then, the following holds (see [10] for the proof).

Proposition 2.8 *Assume further that there are only finite types in $\{K_i\}_{i=1}^M$, i.e. if we define that two K_i's which are similar are equivalent, there are only finite number of equivalence classes in $\{K_i\}_{i=1}^M$. Define $d_s^{\min} = \min_{i=1}^M d_s(K_i)$. Then, there exists $c_{2.1} > 0$ such that the following holds for all $x, y \in \tilde{G}$,*

$$||P_t^{\tilde{\mathcal{E}}}||_{1\to\infty} \leq \begin{cases} c_{2.1}t^{-1}, & \text{for all } t \in (0,1], \\ c_{2.1}t^{-d_s^{\min}/2}, & \text{for all } t \in [1,\infty). \end{cases} \tag{2.4}$$

3 Proof of Proposition 2.6

In this section, we will give a proof of Proposition 2.6. The crucial part is to show 1) for the case $x \vee y \in \partial_i G$ and $x \vee y \in \partial_e G$, where $x \vee y$ means x or y. We adopt completely different methods for the two cases; we use self-similarity and nesting property for the former case and for the latter case, we apply the extension operator used in the trace theory of Besov spaces.

We will first prove 1) for the case $x \vee y \in \partial_i G$. Assumption 2.2 will be used here. For each $f \in C(\mathbf{R}^2)$, let $\|f\|_{\mathrm{Lip}} = sup\{|f(x) - f(y)|/\|x-y\| : x, y \in \mathbf{R}^2\}$ and let Lip $(\mathbf{R}^2) = \{f \in C(\mathbf{R}^2) : \|f\|_{\mathrm{Lip}} < \infty\}$. We now give an important lemma due essentially to Kusuoka ([12]).

Lemma 3.1 *For each Γ_{ij} in (2.1) where $1 \leq i \neq j \leq M$ and for each $K \subset K_i$ which is congruent to $\hat{K}_i$ and $K \cap \Gamma_{ij} \neq \emptyset$, let $H_{\Gamma_{ij},K} : Lip\ (\mathbf{R}^2) \to C(K)$ be a linear operator given by*

$$H_{\Gamma_{ij},K}g(x) = E^x[g(X_{\tau_{\Gamma_{ij}}})], \qquad \textit{for all } x \in K,\ g \in Lip\ (\mathbf{R}^2) \tag{3.1}$$

where $\{X_t\}$ is the Brownian motion on K and $\tau_{\Gamma_{ij}} = \inf\{t \geq 0 : X_t \in \Gamma_{ij}\}$. Then, $H_{\Gamma_{ij},K}g \in \mathcal{F}_K$. Further, there exists $c_{2.2} = c_{2.2}(K) > 0$ such that

$$\mathcal{E}(H_{\Gamma_{ij},K}g, H_{\Gamma_{ij},K}g) \leq c_{2.2}\{\int_{\Sigma_i} (\rho_i L_i \alpha_i^{-2})^{d_{\Gamma_{ij},K}(\omega)} \nu(d\omega)\} \|g\|^2_{Lip} \tag{3.2}$$

holds for any $g \in Lip\ (\mathbf{R}^2)$.

Proof. In the following, we will abbreviate Γ_{ij} to Γ and remove the subscripts i and K. For each $g \in C(K)$, define $h_\Gamma(\cdot : g) : K \to \mathbf{R}$ as follows,

$$h_\Gamma(\pi(\omega) : g) = \begin{cases} E^{\pi(\sigma^m \omega)}[g \circ \Psi_{\omega_1 \cdots \omega_m}(X_{\tau_{V_0}})] & \text{if } d_\Gamma(\omega) = m, \\ g(\pi(\omega)) & \text{if } d_\Gamma(\omega) = \infty, \end{cases} \tag{3.3}$$

for each $\omega \in \Sigma$ (see the Appendix for the notation). It is easy to see that $h_\Gamma(\cdot : g)$ is a well-defined continuous map which is harmonic inside $\Psi_{\omega_1\cdots\omega_m}(K)$ if $d_\Gamma(\omega) = m$, and $h_\Gamma(\cdot : g)|_\Gamma = g|_\Gamma$. Moreover, noting that

$$\mathcal{E}_n(g) = \rho^n \sum_{w \in S^n} \mathcal{E}_0(g \circ \Psi_w) \qquad \text{for all } g \in C(V_n),$$

where we abbreviate $\mathcal{E}_n(g,g)$ to $\mathcal{E}_n(g)$, we can easily see that

$$\mathcal{E}_n(h_\Gamma(\cdot : g)|_{V_n}) = \int_\Sigma \rho^{d_\Gamma(\omega)\wedge n} \cdot L^{d_\Gamma(\omega)\wedge n} \mathcal{E}_0(\{g(\pi([\omega, i]_{d_\Gamma(\omega)\wedge n})); i \in S\})\nu(d\omega), \tag{3.4}$$

where we set $[\omega, i]_l = \omega_1 \cdots \omega_l i i \cdots$. Note also that there exists $c_1 > 0$ such that

$$c_1^{-1}\mathcal{E}_0(u) \le \max\{|u(x) - u(y)|^2 : x, y \in V_0\} \le c_1 \mathcal{E}_0(u) \tag{3.5}$$

for any $u \in C(V_0)$. Using (3.4), (3.5) and the fact $\rho L \alpha^{-2} > 1$ (,which is shown in [1], Proposition 6.30), we have for each $g \in C(K)$ that

$$\begin{aligned} \mathcal{E}_n(h_\Gamma(\cdot : g)|_{V_n}) \;&\le\; c_1 \cdot \{\int_\Sigma (\rho L \alpha^{-2})^{d_\Gamma(\omega)} \nu(d\omega)\} \\ &\times\; \sup_m \{\alpha^m \cdot \max\{|g(x) - g(y)| : x, y \in V_\xi\}; m \ge 0, \xi \in S^m\}^2. \end{aligned}$$

On the other hand, from Assumption 2.2, we see that $A = \int_\Sigma (\rho L \alpha^{-2})^{d_\Gamma(\omega)} \nu(d\omega) < \infty$. We thus obtain

$$\mathcal{E}(h_\Gamma(\cdot : g), h_\Gamma(\cdot : g)) \le c_1 \cdot A \cdot \{\text{diam } K\}^2 \cdot \|g\|_{\text{Lip}}^2,$$

for each $g \in \text{Lip}(\mathbf{R}^2)$. As

$$\mathcal{E}(H_{\Gamma,K} g, H_{\Gamma,K} g) = \inf\{\mathcal{E}(u,u) : \; u \in \mathcal{F}, \; u|_\Gamma = g\} \qquad \text{for all } g \in \text{Lip}(\mathbf{R}^2),$$

we obtain the desired facts. ∎

Using this, we now show 1) of Proposition 2.6 for the case $x \vee y \in \partial_i G \setminus \partial_e G$.

Proposition 3.2 *For each $x \neq y \in \tilde{G}$ where $x \in \partial_i G \setminus \partial_e G$, there exists $f \in \mathcal{D}(\tilde{\mathcal{E}})$ such that $f(x) = 1, f(y) = 0$.*

PROOF. For $x \in \partial_i G \setminus \partial_e G$, denote $\{K_i\}_{i \in I(x)}$ the set of all K_i such that $x \in K_i$. For each K_i $i \in I(x)$, take $m_i \in \mathbf{N}$ such that $\alpha_i^{-m_i-1} \leq e^{-m} < \alpha_i^{-m_i}$ and define $N_m(x)$ as a union of the m_i-complexes which contain x for each $i \in I(x)$. Also, define $N_m^1(x)$ as a union of the m_i-complexes which intersect with $N_m(x)$. We take m suitably large so that $N_m^1(x) \cap \tilde{G} \subset \cup_{i \in I(x)} K_i$, $(\cup_{i \in I(x)} V_0^{(i)}) \cap (N_m^1(x) \setminus N_m(x)) = \emptyset$ and $y \notin N_m^1(x)$. Then, it is enough to prove that there exists $g \in \mathcal{D}(\tilde{\mathcal{E}})$ such that

$$g|_{N_m(x)} = 1, \quad \text{Supp } g \subset N_m^1(x). \tag{3.6}$$

We will now construct $g \in \mathcal{D}(\tilde{\mathcal{E}})$ which satisfies (3.6). Set $g|_{N_m(x)} = 1$ and take an arbitrary connected component of $\Gamma_{ij} \cap (N_m^1(x) \setminus N_m(x))$, $i, j \in I(x)$ which we denote Γ. Denote $a_0 \in N_m(x), a_1 \notin N_m(x)$ end vertices of Γ. Take $f \in \mathrm{Lip}(\mathbf{R}^2)$ so that $f(a_0) = 1, f(a_1) = 0$. Then, by Lemma 3.1, we can construct continuous functions $H_{\Gamma, K_i} f$ and $H_{\Gamma, K_j} f$ on the m_i-complexes of each sides of Γ such that $H_{\Gamma, K_l} f|_\Gamma = f|_\Gamma$ and $\mathcal{E}_{(l)}(H_{\Gamma, K_l} f) < \infty$ for $l = i, j$. We do the same procedure for each connected components of $\Gamma_{ij} \cap (N_m^1(x) \setminus N_m(x))$, $i, j \in I(x)$. Then, using the m-harmonic extension (A.2) for the rest of $N_m^1(x) \setminus N_m(x)$, we can easily extend $\{H_{\Gamma, K_l} f\}_{\Gamma, K_l}$ $(l \in I(x))$ m-harmonically and construct g which satisfies (3.6). By the construction, we see that $g \in \mathcal{D}(\tilde{\mathcal{E}})$. ∎

We next consider the case $x \vee y \in \partial_e G$. As we mentioned, we will apply the extension operator used in the theory of Besov spaces (see [8] for details of the theory). For this purpose, we will briefly explain the construction of an extension operator. It is a slight modification of the operator which extends a function in the Lipschitz (Besov) space on K_i to a function in a Besov space on $\mathbf{R}^N$ ($N = 2$ for our case, but we can argue for all $N \in \mathbf{N}$).

We begin by setting up the Whitney decomposition of the complement of K_i, which has the following properties. It consists of a collection of closed cubes $\{Q_j^{(i)}\}_{j\in\mathbf{N}}$, with mutually disjoint interiors and sides parallel to the axes so that $\mathbf{R}^N \setminus K_i = \cup_j Q_j^{(i)}$. We assume that the sidelength of the cubes is of the form $2^{-M}, M \in \mathbf{Z}$. Denote the center of $Q_j^{(i)}$ by $x_j^{(i)}$, its diameter by $l_j^{(i)}$ and its sidelength by $s_j^{(i)}$. Then $s_j^{(i)} = l_j^{(i)}/\sqrt{n} \in \{2^{-M} : M \in \mathbf{Z}\}$. (In the following, we may omit the superscript ${}^{(i)}$ when there is no confusion.) This decomposition has the following properties,

$$l_j \leq d(Q_j, K_i) \leq 4l_j, \quad Q_j \cap Q_k \neq \emptyset \Rightarrow l_j/4 \leq l_k \leq 4l_j. \tag{3.7}$$

Let $0 < \epsilon < 1/4$ and put $Q_j^* = (1+\epsilon)Q_j$. Note that by the above properties of $\{Q_j\}_j$, each point in $\mathbf{R}^N \setminus K_i$ is contained in at most $N_0(n)$ (which depends only on the Euclidean dimension) cubes Q_j^* and, $Q_j^* \cap Q_k \neq \emptyset$ if and only if $Q_j \cap Q_k \neq \emptyset$. To this decomposition, we associate a partition of unity, consisting of nonnegative functions $\{\varphi_j\}_{j\in\mathbf{N}}$ such that $\varphi_j|_{(Q_j^*)^c} = 0$, $\sum_j \varphi_j(x) = 1$ for all $x \in \mathbf{R}^N \setminus K_i$, and

$$|D^k\varphi_j(x)| \leq A_k(l_j)^{-|k|} \quad \text{for all } x \in \mathbf{R}^N, j \in \mathbf{N}, k \in (\mathbf{N}\cup\{0\})^n, \tag{3.8}$$

for some constant $A_k > 0$ depending only on k. Here, for $k = (k_1, \cdots, k_n)$, we set $D^k = \frac{\partial^{k_1}}{\partial x_1^{k_1}} \cdots \frac{\partial^{k_n}}{\partial x_1^{k_n}}$ and $|k| = k_1 + \cdots k_n$.

We now define the extension operator ξ_{δ_0}. Set $m_j = \mu(B(x_j, 6l_j))^{-1}$. Note that when $l_j = \sqrt{n}2^{-\nu}$ for $\nu \in \mathbf{N}$, then $m_j \leq c_1 2^{\nu d_i}$. Now, for $f \in \mathbf{L}^2(K_i, \mu_i)$, define

$$\xi_{\delta_0} f(x) = \sum_{j\in I_{\delta_0}} \varphi_j(x) m_j \int_{\|t-x_j\|\leq 6l_j} f(t)d\mu_i(t) \quad \text{for all } x \in \mathbf{R}^N \setminus K_i, \tag{3.9}$$

where $\delta_0 > 0$ and

$$I_{\delta_0} \equiv \{j \in \mathbf{N} : s_j \leq c_2\delta_0\}. \tag{3.10}$$

We note that for the usual extension operator, $I \equiv \{j \in \mathbf{N} : s_j \leq 1\}$ is used instead of I_{δ_0}. The concrete value 6 is not important; it is enough to choose sufficiently large number α_0 so that $\mu_i(\{t : ||t - x_j|| \leq \alpha_0 l_j\} \cap K_i)$ is bounded away from 0. Take $f \in C_0(K_i)$. For each fixed $x \in \mathbf{R}^N \setminus K_i$, there are only finite number of φ_j where $\varphi_j(x) \neq 0$ so that $\xi_{\delta_0} f$ is well defined and in $C^\infty(\mathbf{R}^N \setminus K_i)$. Further, by (3.7) and by the definition of I_{δ_0}, $\xi_{\delta_0} f(x) = 0$ if $x \in Q_j, s_j > c_3(\delta_0)$ for some $c_3(\delta_0)$ which depends on c_2 and δ_0. We will take c_2 (which depends only on the dimension of the Euclidean space) small enough so that Supp $\xi_{\delta_0} f$ is in the δ_0-neighborhood of K_i. We thus see that $\xi_{\delta_0} f \in C_b^\infty(\mathbf{R}^N \setminus K_i)$ for $f \in C_0(K_i)$, where $C_b^\infty(\mathbf{R}^N \setminus K_i)$ is a space of infinitely differentiable bounded supported functions on $\mathbf{R}^N \setminus K_i$. In this case, $\xi_{\delta_0} f$ is uniformly continuous on $\mathbf{R}^N \setminus K_i$ and $\lim_{x \to x_0 \in \partial K_i} \xi_{\delta_0} f(x) = f(x_0)$, which can be proved in the same way as in [10] p78, p80. Thus, by defining $\xi_{\delta_0} f(x) = f(x)$ for $x \in K_i$, it holds that $\xi_{\delta_0} f \in C_0(\mathbf{R}^N)$ for each $f \in C_0(K_i)$. It can be also proved by the general trace theory (or for this case as in [10] p79) that $\int_{(K_i)^c} |\nabla(\xi_{\delta_0} f)(x)|^2 dx < \infty$. Noting that Supp $\xi_{\delta_0} f$ is in the δ_0-neighborhood of K_i, we obtain that $\xi_{\delta_0} f \in \mathcal{D}(\tilde{\mathcal{E}})$ for each $f \in C_0(K_i)$.

Using ξ_{δ_0}, we now show 1) of Proposition 2.6 for the case $x \vee y \in \partial_e G \setminus \partial_i G$.

Proposition 3.3 *For each $x \neq y \in \tilde{G}$ where $x \in \partial_e G \setminus \partial_i G$, there exists $f \in \mathcal{D}(\tilde{\mathcal{E}})$ such that $f(x) = 1, f(y) = 0$.*

PROOF. As $x \in \partial_e G \setminus \partial_i G$, there is unique K_i such that $x \in K_i$. Denote $B(x, r)$ a ball in $\mathbf{R}^2$ centered at x and radius r. We take $r, \delta_0 > 0$ small enough so that $U(x, r+\delta_0) \cap G \subset K_i$ and $y \notin U(x, r+\delta_0)$. Using Lemma 2.4 3), we see that there exists $f \in \mathcal{F}_{K_i} \cap C_0(K_i)$ such that $f(x) = 1$ and Supp $f \subset U(x,r) \cap K_i$. Now using the above extension operator, $\xi_{\delta_0} f \in \mathcal{D}(\tilde{\mathcal{E}})$, $(\xi_{\delta_0} f)|_{K_i} = f$ and Supp $f \subset U(x, r+\delta_0)$. Thus $\xi_{\delta_0} f(x) = 1$, $\xi_{\delta_0} f(y) = 0$ and the proof is completed. ∎

End of the proof of Proposition 2.6

We first complete the proof of 1). When $x \vee y \in \tilde{G} \setminus \partial G$, 1) is clear using Lemma 2.4 3) and 4). When x and y are both in ∂G, there are three cases: a) $x \vee y \in \partial_i G \setminus \partial_e G$, b) $x \vee y \in \partial_e G \setminus \partial_i G$, c) $x, y \in \partial_i G \cap \partial_e G$. For the case a) and b), 1) is proved in Proposition 3.2 and Proposition 3.3 respectively. For the case c), denote $\{K_i\}_{i \in I(x)}$ the set of all K_i such that $x \in K_i$. In the same way as Proposition 3.2 (using Lemma 3.1 repeatedly), we can construct $f \in C_0(G)$ such that $f|_{K_i} \in \mathcal{F}_{K_i}$ for all $i \in I(x)$, Supp $f \subset \cup_{i \in I(x)} K_i \setminus \{y\}$ and $f|_{U(x)} = 1$ for some small neighbourhood of x. Now we prepare the Whitney decomposition $\{Q_j\}$ of $(\cup_{i \in I(x)} K_i)^c$, the associated partition of unity $\{\varphi_j\}$ and define $\xi_{\delta_0} f$ in the same way as (3.9) using this $\{Q_j\}$, $\{\varphi_j\}$ and $\mu \equiv \sum_{i \in I(x)} \mu_i$. For $y \in \cup_{i \in I(x)} K_i$, we set $\xi_{\delta_0} f(y) = f(y)$. Then, by taking δ_0 small, we can prove $\xi_{\delta_0} f \in \mathcal{D}(\tilde{\mathcal{E}})$ in the same way as before so that $\xi_{\delta_0} f$ is the desired function.

We next prove 2). For each compact set $L \subset \tilde{G}$, define $I_L = \{i : L \cap K_i \neq \emptyset\}$. Note that $\sharp I_L < \infty$, which is due to Assumption 2.1 2). As each K_i is closed, we can take $\delta_0'(L) > 0$ so that the set of the index of K_i which intersects with $\{y : d(L,y) \leq \delta_0'(L)\}$ is equal to I_L, where d is the Euclidean metric. Now, by the similar way as the proof of 1), there exists $f \in \mathcal{D}(\tilde{\mathcal{E}})$ so that $f|_{L \cap G} = 1$. Now, set $M = L \setminus \cup_{i \in I_L} \{x \in L : f(x) \geq 1/2\}$. Then there exists $g \in C_0^\infty(\mathbf{R}^N)$ so that $g|_M = 1$ and the support of g is in $\{x \in \tilde{G} : d(L,x) \leq \delta_0'(L)\} \setminus G$. Clearly $g \in \mathcal{D}(\tilde{\mathcal{E}})$. Define $h = 2f + g \in \mathcal{D}(\tilde{\mathcal{E}})$. Then, $h|_L \geq 1$. Thus, $\bar{h} \equiv (h \vee 0) \wedge 1$ (which is in $\mathcal{D}(\tilde{\mathcal{E}})$ by the Markovian property of $\tilde{\mathcal{F}}$) is the desired function. ∎

4 Another framework -d-sets floating on $\mathbf{R}^N$-

When we relax Assumption 2.1 and assume Assumption 4.1 instead, then we can construct local regular Dirichlet forms under a wider class of $\{K_i\}_{i=1}^M$ using the same technique we have introduced. In this section, we will briefly discuss it.

Let $K_i \subset \mathbf{R}^N$ $(1 \le i \le M; M$ could be infinite as before) be a closed connected d_i-set for some $0 < d_i \le n$. That is, there exists a Borel measure μ_i whose support is K_i such that

$$c_{4.1} r^{d_i} \le \mu_i(B(x,r)) \le c_{4.2} r^{d_i} \qquad \text{for all } \ x \in K_i, \ r \le c_{4.3}. \tag{4.1}$$

Here $B(x,r)$ is a ball of radius r (centered at x) w.r.t. the Euclidean norm and $c_{4.1}, c_{4.2}, c_{4.3}$ are positive constants which may depend on K_i. We assume the following about the location of $\{K_i\}_{i=1}^M$.

Assumption 4.1 *There exists $\delta_0 > 0$ such that*

$$d(K_i, \cup_{j \ne i} K_j) > \delta_0 \qquad \textit{for all } \ i \in \mathbf{N},$$

where d is the Euclidean distance.

Now, take a set of connected components of $\mathbf{R}^N \setminus \cup_{i=1}^M K_i$, say $\{D_j\}_j$, so that $\tilde{G} \equiv (\cup_{i=1}^M K_i) \cup (\cup_j D_j)$ is a connected closed set. This $\tilde{G}$ is the space we will consider. Set $D = \cup_j D_j$ and define $\mu = m|_D + \sum_{i=1}^\infty \mu_i$. By Assumption 4.1, μ is a well-defined Borel measure.

Examples 4.2 *K_1 is a nested fractal or a Sierpinski carpet, D_1 is a compliment of the convex hull of K_1 and $K_j, D_j = \emptyset$ for all $j \ge 2$. This example is treated in [10]. Especially, when K_1 is the Sierpinski gasket, it is treated also in [7], [14].*

We next give an assumption of the process on each K_i.

Assumption 4.3 *For each $i \in \mathbf{N}$, there is a regular Dirichlet form $(\mathcal{E}_{K_i}, \mathcal{F}_{K_i})$ on $\mathbf{L}^2(K_i, d\mu_i)$ such that*

$$\mathcal{F}_{K_i} \subset Lip(\frac{d_w^{(i)}}{2}, 2, \infty)(K_i) \tag{4.2}$$

for some $d_w^{(i)} \geq 2$ where the Lipschitz space $Lip(d_w^{(i)}/2, 2, \infty)(K_i)$ is a set of $f \in \mathbf{L}^2(K_i, d\mu_i)$ such that

$$\sup_{\nu \in \mathbf{N} \cup \{0\}} \alpha^{\nu(d_w^{(i)}+d_i)} \int\int_{\|x-y\|<c_0\alpha^{-\nu}} |f(x) - f(y)|^2 d\mu_i(x) d\mu_i(y) < \infty \tag{4.3}$$

for some $\alpha > 1, c_0 > 0$.

Remark 4.4 *In [10], it is proved that domains of Dirichlet forms which correspond to Brownian motions on nested fractals and Sierpinski carpets satisfy Assumption 4.3.*

For each D_j, we define a Dirichlet integral

$$\mathcal{E}_{D_j}(u, u) = \frac{1}{2} \int_{D_j} |\nabla u(x)|^2 dx,$$

where ∇u is a distribution function of u on D_j.

We now define a bilinear form $(\tilde{\mathcal{E}}, \mathcal{D}(\tilde{\mathcal{E}}))$ on $\mathbf{L}^2(\tilde{G}, d\mu)$ as follows,

$$\begin{aligned} \tilde{\mathcal{E}}(u, v) &= \sum_{i=1}^{M} \mathcal{E}_{K_i}(u|_{K_i}, v|_{K_i}) + \sum_j \mathcal{E}_{D_j}(u|_{D_j}, v|_{D_j}) \ \text{ for all } u, v \in \mathcal{D}(\tilde{\mathcal{E}}), \\ \mathcal{D}(\tilde{\mathcal{E}}) &= \{u \in C_0(\tilde{G}) : u|_{K_i} \in \mathcal{F}_{K_i} \ \forall\, i, \ \ u|_{D_j} \in W^{1,2}(D_j) \ \forall\, j, \ \ \tilde{\mathcal{E}}(u, u) < \infty\}. \end{aligned}$$

Then, it is easy to check Lemma 2.4 in this framework, too. Denote $\tilde{\mathcal{F}} = \overline{\mathcal{D}(\tilde{\mathcal{E}})}^{\tilde{\mathcal{E}}_{(1)}}$ so that $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ is the smallest extension of $(\tilde{\mathcal{E}}, \mathcal{D}(\tilde{\mathcal{E}}))$. By the similar argument as in the proof of Theorem 2.5, especially that of Proposition 3.3, we have the following.

Theorem 4.5 *$(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ is a local regular Dirichlet form on $\mathbf{L}^2(\tilde{G}, d\mu)$.*

A Appendix

In this appendix, we will briefly summarize nested fractals and Brownian motion on them introduced by Lindstrøm ([13]). See [1],[9], [11] e.t.c. for details.

Let $S = \{1, 2, \cdots, L\}$ $(L < \infty)$ and let $\{\Psi_i\}_{i\in S}$ be *similitude maps* on $\mathbf{R}^N$, i.e., $\Psi_i(x) = \alpha^{-1}U_i x + \beta_i$, $x \in \mathbf{R}^N$ for some unitary maps U_i, $\alpha > 1, \beta_i \in \mathbf{R}^N$. We assume the *open set condition* for $\{\Psi_i\}_{i\in S}$, i.e., there is a non-empty, bounded open set V such that $\{\Psi_i(V)\}_{i\in S}$ are disjoint and $\cup_{i\in S}\Psi_i(V) \subset V$. As $\{\Psi_i\}_{i\in S}$ is a family of contraction maps, there exists a unique non-void compact set $\hat{K}$ such that $\hat{K} = \cup_{i\in S}\Psi_i(\hat{K})$. Before giving the definition of nested fractals, we give some definition and notation. Let F be a set of fixed points of Ψ_i's, $i \in S$ (thus $\sharp F = L$). $x \in F$ is called an essential fixed point if there exist i, j $(i \neq j)$ and $y \in F$ such that $\Psi_i(x) = \Psi_j(y)$. Let V_0 be a set of essential fixed points. Set $V_n = \cup_{x\in V_0} \cup_{i_1,\cdots,i_n\in S} \Psi_{i_1\cdots i_n}(x)$ where $\Psi_{i_1\cdots i_n} \equiv \Psi_{i_1} \circ \cdots \circ \Psi_{i_n}$ and $V_* = \cup_{n\geq 0}V_n$; then $\hat{K} = cl(V_*)$. For $i_1, \cdots, i_n \in S$, we call $\Psi_{i_1\cdots i_n}(V_0)$ n-cell and $\Psi_{i_1\cdots i_n}(K)$ n-complex. For $x, y \in \mathbf{R}^N (x \neq y)$, set $H_{xy} = \{z \in \mathbf{R}^N : |z - x| = |z - y|\}$ and let $U_{xy} : \mathbf{R}^N \to \mathbf{R}^N$ be a symmetric transformation with respect to H_{xy}. Now, $\hat{K}$ is called a (compact) nested fractal if the following holds in addition to the above conditions:

1) $\hat{K}$ is connected, $\sharp V_0 \geq 2$.

2) (Nesting) If $(i_1, \cdots, i_n)$ and $(j_1, \cdots, j_n)$ are distinct elements of S^n, then

$$\Psi_{i_1\cdots i_n}(\hat{K}) \cap \Psi_{j_1\cdots j_n}(\hat{K}) = \Psi_{i_1\cdots i_n}(V_0) \cap \Psi_{j_1\cdots j_n}(V_0).$$

3) (Symmetry) For $x, y \in V_0$ $(x \neq y)$, U_{xy} maps n-cells to n-cells, and it maps any n-cell which contains elements in both sides of H_{xy} to itself for each $n \geq 0$.

From 2), we know that every nested fractal is a finitely ramified fractal. It is known that for each nested fractal, V_0 should be vertices of a regular planar polygon, a d-dimensional tetrahedron or a d-dimensional simplex (see [1], page 71). Set $\Sigma = S^{\mathbf{N}}$ and define a continuous surjective map $\pi : \Sigma \to \hat{K}$ as $\pi(\omega) = \lim_{m\to\infty} \Psi_{\omega_1\cdots\omega_m}(x_0)$ where $x_0 \in V_0$. Let $\sigma : \Sigma \to \Sigma$ be the shift map, i.e. $\sigma w = w_2 w_3 \cdots$ for $w = w_1 w_2 \cdots$.

The Hausdorff dimension of $\hat{K}$ is $\log L / \log \alpha$ $(\equiv d_f)$. A Bernoulli measure $\hat{\mu}$ on $\hat{K}$ with the property $\hat{\mu}(\Psi_{i_1\cdots i_n}(\hat{K})) = L^{-n}$ is a normalized Hausdorff measure.

We will next sumerize how to construct a Dirichlet form on $\hat{K}$. Let $\{l_1, \cdots, l_r\}$ = $\{|x-y| : x, y \in V_0, x \neq y\}$ (where $l_1 < \cdots < l_r$). Set $m_i = \sharp\{y \in V_0 : |x-y| = l_i\}$ (remark that m_i is independent of $x \in V_0$) and let $\mathcal{P} = \{(p_1, \cdots, p_r) : p_1, \cdots, p_r > 0, \sum_{i=1}^r m_i p_i = 1\}$. Now, for $f, g \in l(V_n) \equiv \{f : V_n \to \mathbf{R}\}$ and $(p_1, \cdots, p_r) \in \mathcal{P}$, set

$$\begin{aligned} B_n(f,g) &= \sum_{i_1,\cdots,i_n \in S} \sum_{x,y \in V_0} (f \circ \Psi_{i_1\cdots i_n}(x) - f \circ \Psi_{i_1\cdots i_n}(y)) \\ &\quad \times (g \circ \Psi_{i_1\cdots i_n}(x) - g \circ \Psi_{i_1\cdots i_n}(y)) q_{xy} \end{aligned}$$

(where $q_{xy} = p_i$ if $|x-y| = l_i$, 0 otherwise). Then, it is known that there exists unique $(p_1, \cdots, p_r) \in \mathcal{P}$ and unique $\rho > 1$ such that

$$\rho \cdot \inf\{B_1(g,g) : \ g|_{V_0} = v\} = B_0(v,v) \qquad \text{for all } \ v \in l(V_0). \tag{A.1}$$

In the following we use this $(p_1, \cdots, p_r)$ to define the form. For $f, g \in l(V_n)$, set

$$\hat{\mathcal{E}}_n(f,g) = \rho^n B_n(f,f).$$

Using (A.1) and the nesting property of $\hat{K}$,

$$\hat{\mathcal{E}}_n(f,f) \leq \hat{\mathcal{E}}_{n+1}(f,f) \quad \text{ for all } \ f \in l(V_{n+1})$$

(equality holds when f is harmonic on $V_{n+1} \setminus V_n$). Define

$$\hat{\mathcal{F}} = \{f \in l(V_*) : \lim_{n\to\infty} \hat{\mathcal{E}}_n(f,f) < \infty\}, \quad \hat{\mathcal{E}}(f,g) = \lim_{n\to\infty} \hat{\mathcal{E}}_n(f,g) \quad \text{for all } f,g \in \hat{\mathcal{F}}.$$

Then, for each $f \in \hat{\mathcal{F}}$, there exists unique $P_m f \in \hat{\mathcal{F}}$ such that

$$\hat{\mathcal{E}}(P_m f, P_m f) = \hat{\mathcal{E}}_m(f|_{V_m}, f|_{V_m}), \tag{A.2}$$

which is called a *m-harmonic extension* of $f|_{V_m}$. In order to embed this closed form to $\mathbf{L}^2(\hat{K},\mu)$, we prepare the following.

$$R(p,q)^{-1} = \inf\{\hat{\mathcal{E}}(f,f) : f \in V_*, f(p) = 1, f(q) = 0\} \quad \text{for all } p,q \in V_*,\ p \neq q.$$

This $R(p,q)$ is an effective resistance between p and q. We set $R(p,p) = 0$ for each $p \in V_*$.

Proposition A.1 *1) $R(\cdot,\cdot)$ is a metric on V_*. It can be extended to a metric on $\hat{K}$, (which will be denoted by the same symbol R) and it gives the same topology on $\hat{K}$ as the one from Euclidean metric.*
2) For $p \neq q \in V_*$, $R(p,q) = \sup\{|f(p) - f(q)|^2/\hat{\mathcal{E}}(f,f) : f \in \hat{\mathcal{F}}, f(p) \neq f(q)\}$.

Note that $\rho > 1$ is important for $R(\cdot,\cdot)$ to be a metric on $\hat{K}$. In fact, we have a stronger result on nested fractals. Defining $d_w = \log t_K / \log \alpha$ ($t_K \equiv \rho L$), which is called a walk dimension, we have $R(p,q) \asymp |p-q|^{d_w - d_f}$ ($|\ \cdot\ |$ is a Euclidean metric, $f(x) \asymp g(x)$ means $f(x)/g(x)$ are bounded from above and below by some positive constants). From 2), we have $|f(p) - f(q)|^2 \leq R(p,q)\hat{\mathcal{E}}(f,f)$ for $f \in \hat{\mathcal{F}}, p,q \in V_*$. Therefore $f \in \hat{\mathcal{F}}$ can be extended continuously to $\hat{K}$. By this, we can regard $\hat{\mathcal{F}} \subset C(\hat{K},\mathbf{R}) \subset \mathbf{L}^2(\hat{K},\hat{\mu})$.

Theorem A.2 *$(\hat{\mathcal{E}},\hat{\mathcal{F}})$ is a local regular Dirichlet form on $\mathbf{L}^2(\hat{K},\hat{\mu})$ with the following property.*

$$|f(p) - f(q)|^2 \leq R(p,q)\hat{\mathcal{E}}(f,f) \qquad \textit{for all } f \in \hat{\mathcal{F}}, \textit{and } p,q \in \hat{K} \tag{A.3}$$

$$\hat{\mathcal{E}}(f,g) = \rho \sum_{i \in S} \hat{\mathcal{E}}(f \circ \Psi_i, g \circ \Psi_i) \qquad \textit{for all } f, g \in \hat{\mathcal{F}} \tag{A.4}$$

Further, for $\beta > 0$, $\hat{\mathcal{E}}_{(\beta)}$ admits a positive symmetric continuous reproducing kernel.

By the general theory ([2]), there is a one to one correspondence between a local regular Dirichlet form on $\mathbf{L}^2(\hat{K}, \hat{\mu})$ and a $\hat{\mu}$-symmetric diffusion process on $\hat{K}$ except some exceptional set of starting points. In this case, thanks to (A.3), we can prove the Feller property of the process so that the one to one correspondence holds without any ambiguity of the starting points. We will denote $\{\hat{X}_t\}_{t \geq 0}$ the diffusion process corresponding to $(\hat{\mathcal{E}}, \hat{\mathcal{F}})$. Roughly saying, this process is constructed from the random walk $\hat{X}_n$ on V_n (whose transition probability is given by $(p_1, \cdots, p_r)$) by multiplying t_K^n to the time (,which is $\hat{X}_n([t_K^n t])$) and taking $n \to \infty$. It is known that any self-similar Feller diffusion process which is invariant under local symmetric transformations on $\hat{K}$ is a constant time change of this process, so that we call this process Brownian motion on $\hat{K}$.

Define $d_s = 2 \log L / \log t_K$ which is called a spectral dimension and $d_w^R = d_w/(d_w - d_f)$ which is a walk dimension w.r.t. the resistance metric $R(\cdot, \cdot)$.

Theorem A.3 *Brownian motion on $\hat{K}$ has a jointly continuous transition density (heat kernel) $\hat{p}_t(x,y)$ $t > 0, x, y \in \hat{K}$. Further, there exist $d_c > 0$ and $c_{A.1}, \cdots, c_{A.4}$ such that*

$$\begin{aligned} c_{A.1} t^{-d_s/2} \exp\left(-c_{A.2}\left(\frac{R(x,y)^{d_w^R}}{t}\right)^{\frac{d_c}{d_w^R - d_c}}\right) &\leq \hat{p}(t,x,y) \\ &\leq c_{A.3} t^{-d_s/2} \exp\left(-c_{A.4}\left(\frac{R(x,y)^{d_w^R}}{t}\right)^{\frac{d_c}{d_w^R - d_c}}\right), \end{aligned}$$

for all $0 < t < 1$ and all $x, y \in \hat{K}$.

Theorem A.4 *([10])*

$$\hat{\mathcal{F}} = Lip(\frac{d_w}{2}, 2, \infty)(\hat{K}), \tag{A.5}$$

where the Lipschitz space $Lip(d_w/2, 2, \infty)(\hat{K})$ *is a set of* $f \in \mathbf{L}^2(\hat{K}, \hat{\mu})$ *such that*

$$\sup_{\nu \in \mathbf{N} \cup \{0\}} \alpha_0^{\nu(d_w+d_f)} \int \int_{\|x-y\|<c_0\alpha_0^{-\nu}} |f(x) - f(y)|^2 d\hat{\mu}(x) d\hat{\mu}(y) < \infty \tag{A.6}$$

for some $\alpha_0 > 1, c_0 > 0$.

Note that it is easy to see that in (A.6), different values on the constants c_0 and α_0 give equivalent spaces as long as the former is positive and the latter is greater than 1. It is known that when $d_w/2 \notin \mathbf{Z}$, this Lipschitz space corresponds to (a subspace of) the Besov space $B^{2,\infty}_{d_w/2}(\hat{K})$ (see [8] Chapter V Proposition 3 and [6] Proposition 1).

Now assume without loss of generality that $\Psi_1(x) = \alpha^{-1}x$. Then, an unbounded nested fractal K is constructed as $K = \cup_{n=1}^{\infty} \alpha^n \hat{K}$. The local regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on K, whose restriction to $\hat{K}$ is $\hat{\mathcal{E}}$, can be constructed on $\mathbf{L}^2(K, \mu)$ (where μ is a Bernoulli measure on K so that $\mu|_{\hat{K}} = \hat{\mu}$) as follows. Set $\hat{K}_{<l>} = \alpha^l \hat{K}$ and define $\sigma_l : l(\hat{K}_{<l>}) \to l(\hat{K})$ by $\sigma_l f(x) = f(\alpha^l x) = f \circ \Psi_1^{-l}(x)$ for all $x \in \hat{K}$. Set $\hat{\mathcal{F}}_{\hat{K}_{<l>}} = \sigma_{-l}\hat{\mathcal{F}}$ and $\hat{\mathcal{E}}_{\hat{K}_{<l>}}(f, g) = \rho^{-l}\hat{\mathcal{E}}(\sigma_l f, \sigma_l g)$ for all $f, g \in \hat{\mathcal{F}}_{\hat{K}_{<l>}}$. It is easy to see

$$\hat{\mathcal{E}}_{\hat{K}_{<l-1>}}(f|_{\hat{K}_{<l-1>}}, f|_{\hat{K}_{<l-1>}}) \le \hat{\mathcal{E}}_{\hat{K}_{<l>}}(f, f) \qquad \text{for all } \ f \in \hat{\mathcal{F}}_{\hat{K}_{<l>}}. \tag{A.7}$$

Define

$$\begin{aligned}
\mathcal{D}_K &= \{f \in C_0(K) : f|_{\hat{K}_{<l>}} \in \mathcal{F}_{\hat{K}_{<l>}} \forall l \in \mathbf{N}, \ \lim_{l \to \infty} \hat{\mathcal{E}}_{\hat{K}_{<l>}}(f|_{\hat{K}_{<l>}}, f|_{\hat{K}_{<l>}}) < \infty\}, \\
&\quad \mathcal{E}(f, g) = \lim_{l \to \infty} \hat{\mathcal{E}}_{\hat{K}_{<l>}}(f|_{\hat{K}_{<l>}}, g|_{\hat{K}_{<l>}}) \qquad \text{for all } \ f, g \in \mathcal{D}_K.
\end{aligned}$$

It is easy to show that $(\mathcal{E}, \mathcal{D}_K)$ is closable in $\mathbf{L}^2(K, \mu)$ by using $(A.7)$. Denote $\mathcal{F} = \overline{\mathcal{D}_K}^{\mathcal{E}_{(1)}}$ so that $(\mathcal{E}, \mathcal{F})$ is the smallest extension of $(\mathcal{E}, \mathcal{D}_K)$. Then we can define the resistance metric $R(\cdot, \cdot)$ in the same way and we have the following.

Theorem A.5 *$(\mathcal{E},\mathcal{F})$ is a local regular Dirichlet form on $\mathbf{L}^2(K,\mu)$ which satisfies (A.3) and the following scaling property,*

$$\mathcal{E}(f,g)=\lambda\mathcal{E}(f\circ\Psi_1,g\circ\Psi_1)\ \textit{for all}\ \ f,g\in\mathcal{F}.$$

Further, for $\beta>0$, $\mathcal{E}_{(\beta)}$ admits a positive symmetric continuous reproducing kernel.

We call the corresponding diffusion process Brownian motion on K. Theorem A.3 holds for the heat kernel on K for $0<t<\infty$. Similarly to Theorem A.4, we have $\mathcal{F}=\tilde{\mathrm{Lip}}(\frac{d_w}{2},2,\infty)(K)$, where $\tilde{\mathrm{Lip}}(d_w/2,2,\infty)(K)$ is a set of $f\in\mathbf{L}^2(K,\mu)$ such that

$$\sup_{\nu\in\mathbf{Z}}\alpha^{\nu(d_w+d_f)}\int\int_{||x-y||<c_0\alpha_0^{-\nu}}|f(x)-f(y)|^2d\mu(x)d\mu(y)<\infty \tag{A.8}$$

for some $\alpha_0>1,c_0>0$.

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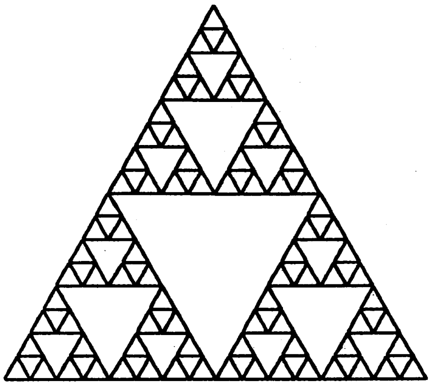

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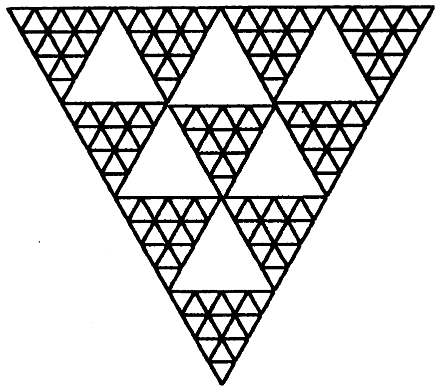

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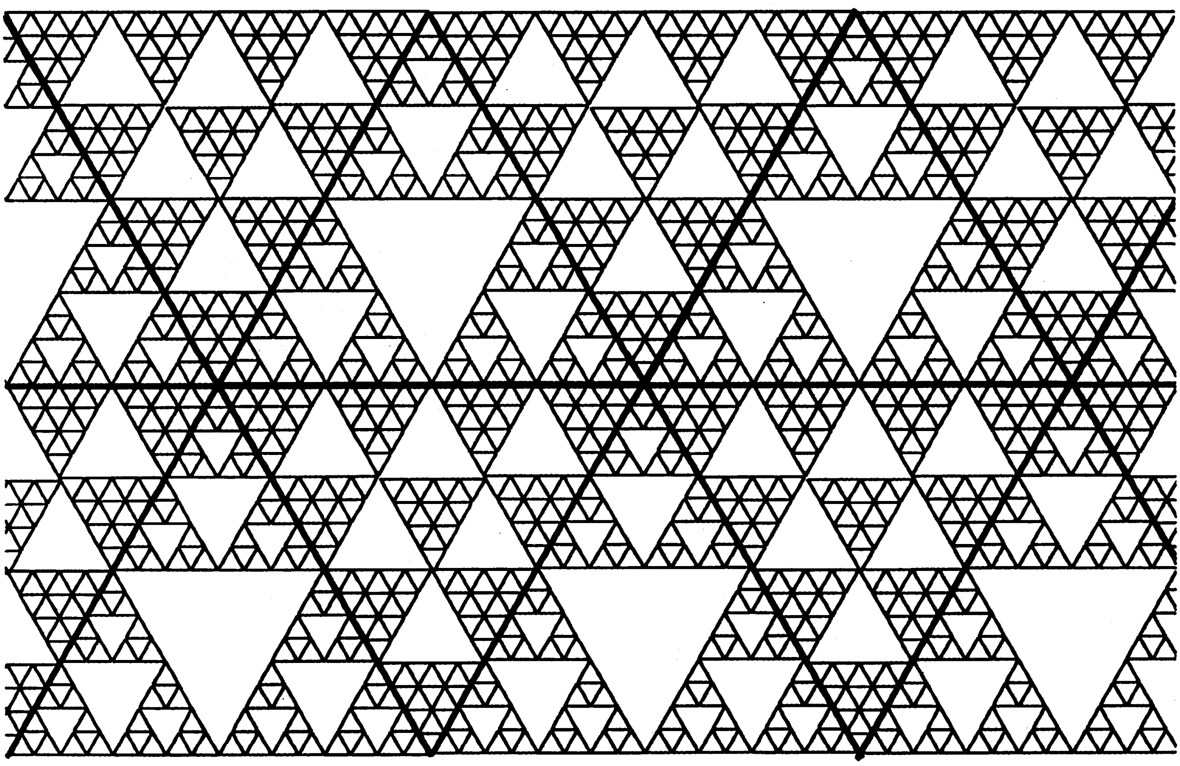

Figure 1: An example of the fractal field (gasket tiling)

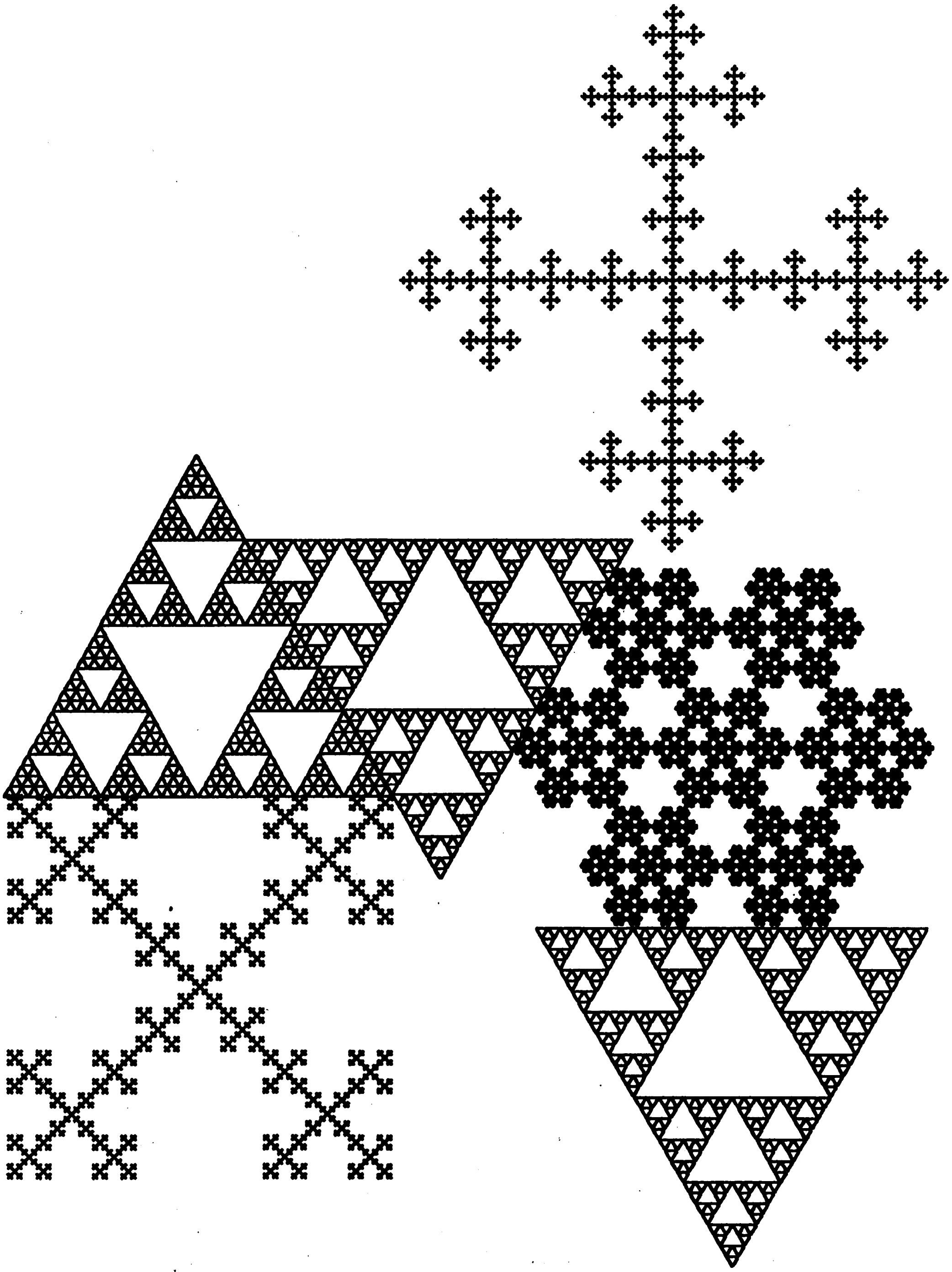

Figure 2: An example of the fractal field